Quiz 2: Chapter 1
Exam 1 Review

Answer the following questions. *You must show your work to receive full credit.*

Section 1.1. Verify that $y = x\cos x$ is a solution to the differential equation

$$y' + y\tan x = \cos x.$$

$y = x\cos x$

$y' = \cos x - x\sin x$

$\Rightarrow \cos x - x\sin x + x\cos x \cdot \tan x$

$= \cos x - x\sin x + x\sin x$

$= \cos x$ ✓

Section 1.2. A diesel car gradually speeds up so that for the first 10s its acceleration is given by

$$\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 seconds and its velocity at that time.

$v(t) = \int (0.12)t^2 + (0.6)t\,dt = (0.04)t^3 + (0.3)t^2 + C$ ft/s

$v(0) = 0 = C$

$x(t) = \int (0.04)t^3 + (0.3)t^2\,dt = (0.01)t^4 + (0.1)t^3 + C$ ft

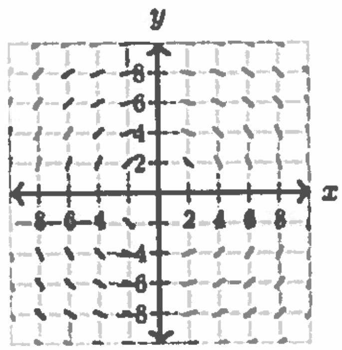
$x(0) = 0 = C$

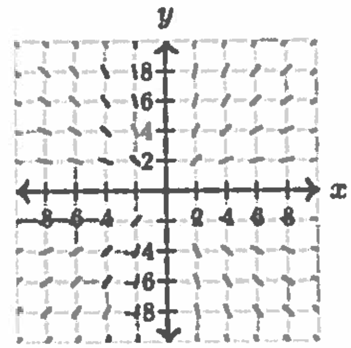
So $x(10) = (0.01)(10)^4 + (0.1)10^3 = 200$ ft

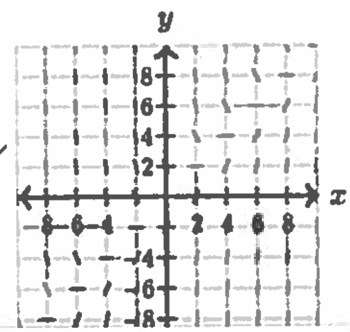
and

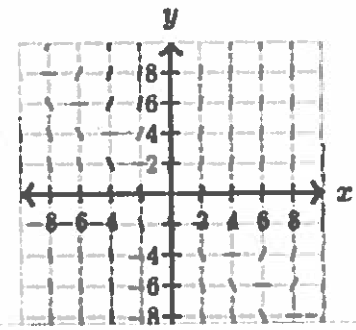
$v(10) = (0.04)10^3 + (0.3)10^2 = 70$ ft/s

Section 1.3. Which slope field is generated by the differential equation $\frac{dy}{dx} = x - y$?

- ○ 

- ○ 

- ☒ 

- ○ 

Section 1.4. Find the general solution to the differential equation

$$2\sqrt{x}\frac{dy}{dx} = \cos^2 y.$$

$$\int \frac{dy}{\cos^2 y} = \int \frac{dx}{2\sqrt{x}}$$

$$\tan y = \sqrt{x} + C$$

$$Y = \tan^{-1}(\sqrt{x} + C)$$

Section 1.5. A tank initially contains 60 gallons of pure water. Brine containing 1 lb of salt per gallon enters the tank at a rate of 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min.

(a) (2 points) When is the tank empty?

(b) (6 points) Find the amount of salt in the tank after t minutes.

(c) (2 points) What is the maximum amount of salt ever in the tank?

(a) $V(t) = 60 + 2t - 3t = 60 - t$ gallons. Empty in 60 minutes.

(b) $x' + \frac{3}{60-t} x = 2 \cdot 1$, $P(t) = e^{\int \frac{3}{60-t} dt} = e^{-3\ln(60-t)} = (60-t)^{-3}$

$$(60-t)^{-3} x = \int 2 \cdot (60-t)^{-3} dt = (60-t)^{-2} + C$$

$$\Rightarrow \quad x = (60-t) + C \cdot (60-t)^3$$

$$x(0) = 0 = 60 + C \cdot 60^3 \Rightarrow C = -60^{-2} = \frac{-1}{3600}$$

and $$x = (60-t) - \frac{(60-t)^3}{3600}$$

(c) $x' = -1 + \frac{3}{3600}(60-t)^2 \Rightarrow t = 60 \pm \sqrt{1200}$

Only $t = 60 - \sqrt{1200}$ makes sense. It is a max.

$$x(60 - \sqrt{1200}) = \sqrt{1200} - \frac{1200^{3/2}}{3600} \approx 23.09 \text{ lbs of salt.}$$

Section 1.6a. Find the general solution to the differential equation

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0.$$

$$\int 1 + ye^{xy}\,dx = x + e^{xy} + C(y)$$

$$\int 2y + xe^{xy}\,dy = y^2 + e^{xy} + C(x).$$

So

$$F(x,y) = e^{xy} + x + y^2 = C.$$

Section 1.6b. Consider a rabbit population satisfying the logistic equation

$$\frac{dP}{dt} = 2P - (0.005)P^2.$$

If the initial population is 120 rabbits, how many months does it take for $P(t)$ to reach 95% of its limiting population M?

$$\frac{dP}{dt} = 2P - (0.005)P^2 = (0.005)P\cdot(400 - P)$$

<u>Logistic Eqn Solution</u>

$$P(t) = \frac{120 \cdot 400}{120 + 280e^{-2t}}$$

$$380 = \frac{48000}{120 + 280e^{-2t}}$$

$$126.32 = 120 + 280e^{-2t}$$

$$6.32 = 280e^{-2t}$$

$$0.0226 = e^{-2t}$$

$$t = \frac{\ln(0.0226)}{-2} \approx 1.896 \text{ years.}$$